Example Involving Probability Distributions & Expected Value

Example
If the table below represents the PDF of the random variable X, find $E(X)$ (the expected value of X).

<table>
<thead>
<tr>
<th>$X$</th>
<th>-1</th>
<th>3</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X)$</td>
<td>0.3</td>
<td>0.2</td>
<td>0.5</td>
</tr>
</tbody>
</table>

$E(X) = (-1) \cdot P(-1) + 3 \cdot P(3) + 5 \cdot P(5)$

$E(X) = (-1) \cdot (0.3) + 3(0.2) + 5(0.5)$

$E(X) = -0.3 + 0.6 + 2.5$

$E(X) = 2.8$
Example Involving Probability Distributions & Expected Value

Example
Find the expected value for the random variable:

<table>
<thead>
<tr>
<th>X</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(X)</td>
<td>0.1</td>
<td>0.3</td>
<td>0.5</td>
<td>0.1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>X</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(X)</td>
<td>0.1</td>
<td>0.3</td>
<td>0.5</td>
<td>0.1</td>
</tr>
</tbody>
</table>

| X · P(X) | 0.2 | 0.9 | 2.0 | 0.5 |

\[ E(X) = 0.2 + 0.9 + 2.0 + 0.5 \]
\[ E(X) = 3.6 \]
Example Involving Probability Distributions & Expected Value

**Example**

A business bureau gets complaints as shown in the following table. Find the expected number of complaints per day.

<table>
<thead>
<tr>
<th>X</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(X)</td>
<td>0.04</td>
<td>0.11</td>
<td>0.21</td>
<td>0.33</td>
<td>0.19</td>
<td>0.12</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>X</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(X)</td>
<td>0.04</td>
<td>0.11</td>
<td>0.21</td>
<td>0.33</td>
<td>0.19</td>
<td>0.12</td>
</tr>
</tbody>
</table>

| X·P(X) | 0   | 0.11 | 0.42 | 0.99 | 0.76 | 0.60 |

\[ E(X) = 0 + 0.11 + 0.42 + 0.99 + 0.76 + 0.60 \]

\[ E(X) = 2.88 \]
Example Involving Probability Distributions & Expected Value

Example

Find the expected value for the random variable $X$ having this probability function:

<table>
<thead>
<tr>
<th>$X$</th>
<th>9</th>
<th>11</th>
<th>13</th>
<th>15</th>
<th>17</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X)$</td>
<td>0.1</td>
<td>0.3</td>
<td>0.1</td>
<td>0.3</td>
<td>0.2</td>
</tr>
<tr>
<td>$X \cdot P(X)$</td>
<td>0.9</td>
<td>3.3</td>
<td>1.3</td>
<td>4.5</td>
<td>3.4</td>
</tr>
</tbody>
</table>

$$E(X) = 0.9 + 3.3 + 1.3 + 4.5 + 3.4 = 13.4$$
Example Involving Probability Distributions & Expected Value

Example
A contractor is considering a sale that promises a profit of $23,000 with a probability of 0.7 or a loss (due to bad weather, strikes, etc.) of $13,000 with a probability of 0.3. What is the expected profit?

<table>
<thead>
<tr>
<th>X</th>
<th>23000</th>
<th>-13000</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(X)</td>
<td>0.7</td>
<td>0.3</td>
</tr>
<tr>
<td>X·P(X)</td>
<td>16100</td>
<td>-3900</td>
</tr>
</tbody>
</table>

\[ E(X) = 16100 + (-3900) = 12200 \]

Expected profit is $12,200
Example Involving Probability Distributions & Expected Value

Example
John buys one $5 raffle ticket (out of a total of 500 sold in all). If a single randomly selected winner gets a $100 prize, what are John’s expected winnings?

Let \( X \) = the amount of money won (+) or lost (-)
\( P(X) \) = the probability of winning or losing that amount

<table>
<thead>
<tr>
<th>( X )</th>
<th>(-$5)</th>
<th>$95</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(X) )</td>
<td>( \frac{499}{500} )</td>
<td>( \frac{1}{500} )</td>
</tr>
<tr>
<td>( X \cdot P(X) )</td>
<td>( \frac{-2495}{500} )</td>
<td>( \frac{95}{500} )</td>
</tr>
</tbody>
</table>

\[
E(X) = \frac{-2495}{500} + \frac{95}{500} = \frac{-2400}{500} = -$4.80
\]
Example Involving Probability Distributions & Expected Value

A raffle offers a first prize of $1000, 2 second prizes of $300 and 20 third prizes of $10 each. If 1000 tickets are sold at 50 cents each, find the expected winnings for a person buying 1 ticket.

Let $X$ = expecting winnings

4 possibilities:

LOSE $\rightarrow X = -$0.50

3rd $\rightarrow X = $9.50 (= $10.00-$0.50 for ticket)

2nd $\rightarrow X = $299.50 (= $300.00-$0.50 for ticket)

1ST $\rightarrow X = $999.50 (= $1000.00-$0.50 for ticket)

<table>
<thead>
<tr>
<th>$X$</th>
<th>-$0.50</th>
<th>$9.50</th>
<th>$299.50</th>
<th>$999.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X)$</td>
<td>$\frac{997}{1000}$</td>
<td>$\frac{20}{1000}$</td>
<td>$\frac{2}{1000}$</td>
<td>$\frac{1}{1000}$</td>
</tr>
</tbody>
</table>

$X \cdot P(X)$

| $X \cdot P(X)$ | Notice that you can continue from here to get the answer but the calculations are complex. Below is a suggestion for making it much easier.

ALTERNATE (easier way): Ignore the cost of the ticket until the end and then, at the end, subtract it off. So, if we temporarily ignore the cost of the ticket:

<table>
<thead>
<tr>
<th>$X$</th>
<th>-$0</th>
<th>$10</th>
<th>$300</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X)$</td>
<td>$\frac{997}{1000}$</td>
<td>$\frac{20}{1000}$</td>
<td>$\frac{2}{1000}$</td>
<td>$\frac{1}{1000}$</td>
</tr>
<tr>
<td>$X \cdot P(X)$</td>
<td>0</td>
<td>$\frac{600}{1000}$</td>
<td>$\frac{1000}{1000}$</td>
<td>$\frac{1000}{1000}$</td>
</tr>
</tbody>
</table>

$$E(X) = 0 + \frac{200}{1000} + \frac{600}{1000} + \frac{1000}{1000} = \frac{1800}{1000} = $1.80$$

Now subtract off the ticket cost that we earlier ignored: $1.80-0.50=\underline{1.30}$ (expected winnings)
Example Involving Probability Distributions & Expected Value

Suppose you buy 1 ticket for $1 out of a lottery of 1000 tickets where the prize for the winning ticket is to be $500. What are your expected winnings?

Let $X = \text{expected winnings}$

2 possibilities: LOSE $\Rightarrow X = -$1 or WIN $\Rightarrow X = $499 ($500 - $1 for ticket)

<table>
<thead>
<tr>
<th>$X$</th>
<th>-$1</th>
<th>$499$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X)$</td>
<td>$\frac{999}{1000}$</td>
<td>$\frac{1}{1000}$</td>
</tr>
<tr>
<td>$X \cdot P(X)$</td>
<td>$\frac{-999}{1000}$</td>
<td>$\frac{495}{1000}$</td>
</tr>
</tbody>
</table>

$$E(X) = \frac{-999}{1000} + \frac{495}{1000} = \frac{-500}{1000}$$

$E(X) = -$0.50
Two cards are drawn with replacement from a deck of cards and the number of black cards is noted. If $X$ is the random variable representing the number of black cards drawn, construct the PDF.

**Solution:** Let's use a tree diagram to aid us. (R-red, B-black)

By the Multiplication Principle:
- $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$
- $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$
- $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$
- $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

Note:
- $\frac{2}{6}$ Red & $\frac{2}{6}$ Black
- So $P(R) = P(B) = \frac{1}{2}$

So, the sample space is:
\[
\{BB, BR, RB, RR\}
\]

Remember that $X = \#$ of black cards drawn so...

- $P(X = 0) = P(RR) = \frac{1}{4}$
- $P(X = 1) = P(BR) + P(RB) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$
- $P(X = 2) = P(BB) = \frac{1}{4}$

So the PDF is:
\[
\begin{array}{c|cccc}
X & 0 & 1 & 2 \\
P(X) & \frac{1}{4} & \frac{1}{2} & \frac{1}{4}
\end{array}
\]
Two cards are drawn with replacement from a deck of cards and the number of black cards is noted. If \( X \) is the random variable representing the number of black cards drawn, construct the PDF.

**Alternate Solution:** Let's use a tree diagram to aid us (R-red, B-black)

\[
\begin{array}{c}
\text{B: BB} \\
\text{R: BR} \\
\text{R: RB} \\
\text{R: RR}
\end{array}
\]

If we are confident that we are working with an equally-likely sample space, we can avoid having to use probabilities on the tree diagram branches and instead just use the tree diagram to generate this equally-likely sample space.

**Equally-Likely Sample Space:** \( \{ \text{BB, BR, RB, RR} \} \)

(Notice that as long as we know that we have an equally-likely sample space, we don't need the "branch" probabilities that we calculated using the previous solution method.) Instead we just need to identify the number of black cards in each of the 4 equally-likely elements.

\[
\begin{array}{c|cccc}
\text{BB} & \text{BR} & \text{RB} & \text{RR} \\
0 & 1 & 1 & 0
\end{array}
\]

\[
\begin{align*}
P(X=0) &= \frac{1}{4} \\
P(X=1) &= \frac{1}{4} \\
P(X=2) &= \frac{1}{4} \\
P(X=3) &= \frac{1}{4}
\end{align*}
\]

So, the PDF is

\[
\begin{array}{c|cccc}
X & 0 & 1 & 2 & 3 \\
P(X) & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}
\end{array}
\]